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QUASILINEARIZATION, BOUNDARY-VALUE PROBLEMS AND LINEAR PROGRAMMING

R. E. Bellman, H. H. Kagiwada and R. E. Kalaba

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The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA • 90406

PREFACE AND SUMMARY

In earlier papers we have shown that many problems in orbit determination, system identification, vector cardiology, and so on can be considered to be nonlinear multi-point boundary-value problems. Quasilinearization offers an effective computational method of solution. When the number of conditions on the solution exceeds the number of degrees of freedom, we have made extensive use of the method of least squares.

In this note we show that the problem of minimizing the maximum deviation can be solved using the quasilinearization format and employing linear programming at each stage of the calculation. In the case where the observational errors are all of approximately the same magnitude, or are all of the same relative magnitudes, this procedure seems to have substantial advantages over the method of least squares.

I. INTRODUCTION

In a series of papers we have shown that the method of quasilinearization provides an effective computational tool for the solution of a wide class of nonlinear two-point and multi-point boundary-value problems.[1-4] Various applications to the solution of Euler equations,[5] orbit determination,[6] partial differential equations,[7] vectorcardiology[8] and system identification[9] have been made. As a rule, when the number of conditions which the solution of a system of differential equations must satisfy exceeds the number of available constants, we have employed the method of least squares. In this Memorandum we shall show how we can use instead the minimax criterion in conjunction with standard linear programming codes,[10] and thereby derive certain advantages.

II. FORMULATION

Consider an N-dimensional vector x(t) which is a solution of the nonlinear system of differential equations

$$\dot{x} = f(x) \ . \tag{1}$$

Suppose that we wish to determine a solution for which

$$(\alpha_i \ , \ x(t_i)) = \beta_i, \qquad i = 1,2,\ldots,M \geq N \ . \tag{2}$$

For example, through observations of a process we might know the values of certain components of x at various times. Due to experimental errors in the observations we would not expect that any solution of Eq. (1) would satisfy all of the conditions in Eq. (2). A

standard approach is then to seek a solution which minimizes a weighted sum of squares of deviations S,

$$S = \sum_{i=1}^{M} w_i\{(\alpha_i , x(t_i)) - \beta_i\}^2 . \tag{3}$$

As is shown in the references, this can be very effective.

Alternatively, let us consider determining a solution of Eq. (1) for which the maximum of the difference of the absolute values between $(\alpha_i, x(t_i))$ and β_i is as small as possible.

III. SKETCH OF METHOD

Let γ be an initial approximation to the initial vector and let z(t), determined numerically by the equations

$$\dot{z} = f(z) , \qquad z(0) = \gamma , \tag{4}$$

be an initial approximation to the optimal function x(t) on $0 \le t \le t_M$. To obtain the next approximation we linearize in the usual way

$$\dot{w} = f(z) + J(z)(w-z) , \tag{5}$$

where J(z) is the Jacobian matrix whose i^{th} row and j^{th} column, J_{ij}, is

$$J_{ij} = \partial f_i/\partial z_j . \tag{6}$$

We regard z(t) and J(z(t)) as computationally known. Equation (5) is a linear system in the unknown vector w(t). Its solution is representable in the form

$$w(t) = p(t) + \sum_{i=1}^{N} c_i \, h_i(t) \ , \tag{7}$$

where the particular solution $p(t)$ is obtained numerically on $0 \le t \le t_M$ as the solution of the initial value problem

$$\dot{p} = f(z) + J(z)(p-z) \ , \quad p(0) = 0 \ , \tag{8}$$

and the homogeneous vector $h_i(t)$ is the solution of the equations

$$\dot{h}_i = J(z)h_i \ , \qquad h_i(0) = \delta_i \ , \tag{9}$$

where the vector δ_i has all components zero except the i^{th}, which is unity. We determine the constants c_i, which represent the initial values of w, $w(0)$, by the condition that they should minimize the maximum of the absolute value of the difference between $(\alpha_i, w(t_i))$ and β_i. Mathematically we wish to determine the unknowns c_i, ϵ_i, and ϵ so that we minimize ϵ, where we have the 3M linear inequalities,

$$\pm \left[\left(\alpha_i, p(t_i) + \sum_{j=1}^{N} c_j \, h_j(t_i) \right) - \beta_i \right] \le \epsilon_i \ ,$$

$$\epsilon_i \le \epsilon \ , \tag{10}$$

$$i = 1,2,\ldots,M \ .$$

The quantities $p(t_i)$ and $h_j(t_i)$ are known computationally, and β_i is an observed value. This is a linear programming problem[10] for which standard codes for effecting the numerical solution for $M \le 10^2$ are available. Once $c_1, c_2, \ldots, c_N$ are determined, the vector $w(t)$ is

produced through use of Eq. (7) for $0 \leq t \leq t_M$. Then the procedure is repeated to obtain the next approximation.

IV. EXAMPLES--IDENTIFICATION OF A NONLINEAR SYSTEM

Suppose that a system is undergoing a process described by the Van der Pol equation[9,11]

$$\ddot{x} + \lambda(x^2-1)\dot{x} + x = 0 . \tag{11}$$

We observe experimentally that

$$x(4.0) \cong - 1.21184 \tag{12}$$

$$x(4.1) \cong - 1.12551 \tag{13}$$

$$x(4.2) \cong - 1.04186 . \tag{14}$$

We wish to determine the unknown system parameter λ and the unknown initial velocity, $\dot{x}(4)$. We replace the Eq. (11) with the system

$$\dot{x} = u \tag{15}$$

$$\dot{u} = - \lambda(x^2-1)u - x \tag{16}$$

$$\dot{\lambda} = 0 . \tag{17}$$

As initial approximations suppose that we use

$$x(4.0) \cong - 1.2 \tag{18}$$

$$u(4.0) \cong 0.7 \tag{19}$$

$$\lambda(0) \cong 1.0 . \tag{20}$$

Then about 50 sec of calculation on an IBM 7090, using the above method, produced the values below.

Table 1

NUMERICAL RESULTS

Iteration	x(4.0)	u(4.0)	λ
0	-1.20000	.70000	1.00000
1	-1.21184	.68767	2.06717
2	-1.21184	.68739	1.99828
True Values	-1.21184	.68740	2.00000

V. DISCUSSION

Many other numerical experiments have been conducted, a majority with more complex systems than in the example discussed above. Generally speaking, we have found that the method of least squares is preferable if the observational errors are Gaussian. If, however, the errors are all of about the same magnitude or all of about the same per cent deviation, then the minimax procedure leads to superior estimates. In many instances, though, where we should expect the method of least squares to be better, we have found the minimax estimators to be superior. Much remains to be done to understand the relative advantages and disadvantages of these approaches.

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